

$f(R)$ gravity with torsion: the metric-affine approach

S. Capozziello¹, R. Cianci², C. Stornaiolo¹, S. Vignolo²

¹ *Dipartimento di Scienze Fisiche, Università “Federico II” di Napoli and INFN Sez. di Napoli, Compl. Univ. Monte S. Angelo Ed. N, via Cinthia, I- 80126 Napoli (Italy) and*

² *DIPTeM Sez. Metodi e Modelli Matematici, Università di Genova, Piazzale Kennedy, Pad. D - 16129 Genova (Italy)*

The role of torsion in $f(R)$ gravity is considered in the framework of metric-affine formalism. We discuss the field equations in empty space and in presence of perfect fluid matter taking into account the analogy with the Palatini formalism. As a result, the extra curvature and torsion degrees of freedom can be dealt as an effective scalar field of fully geometric origin. From a cosmological point of view, such a geometric description could account for the whole Dark Side of the Universe.

PACS numbers: 04.20.Cv, 04.20.+Fy, 04.20.Gz, 98.80.-k

I. INTRODUCTION

In the last thirty years, some shortcomings came out in the Einstein General Relativity (GR) and several investigations started in order to study if alternative approaches to gravitational interaction are possible and self-consistent. Such issues come from Cosmology and Quantum Field Theory. In the first case, the presence of Big Bang singularity, flatness and horizon problems [1] led to the result that Standard Cosmological Model [2], based on GR and Standard Model of particle physics, is inadequate to describe the Universe at extreme regimes. On the other hand, GR does not work for a quantum description of spacetime. Due to this facts and to the lack of a Quantum Gravity theory, alternative theories of gravity have been pursued in order to attempt, at least, a semi-classical scheme where GR and its positive results could be recovered.

A fruitful approach has been that of *Extended Theories of Gravity* (ETG) which have become a sort of paradigm in the study of gravitational interaction. They are essentially based on corrections and enlargements of the Einstein theory. The paradigm consists in adding higher-order curvature invariants and non-minimally coupled scalar fields into dynamics resulting from the effective action of Quantum Gravity [3, 4].

Other motivations to modify GR come from the issue to recover Mach’s principle [5]. This principle states that the local inertial frame is determined by some average of the motion of distant astronomical objects [6], so that gravitational coupling can be scale-dependent and related to some scalar field. This viewpoint leads to assume a varying gravitational coupling. As a consequence, the concepts of “inertia” and equivalence principle have to be revised [5, 7, 8, 9].

Furthermore, every unification scheme as Superstrings, Supergravity or Grand Unified Theories, takes into account effective actions where non-minimal couplings to the geometry or higher-order terms in the curvature invariants come out. Such contributions are due to one-loop or higher-loop corrections in the high-curvature regimes. In particular, this scheme has been adopted in order to deal with quantization in curved spacetimes and, as a result, the interactions between quantum scalar fields, the gravitational self-interactions and the background geometry yield correction terms in the Hilbert-Einstein Lagrangian [10].

Moreover, it has been realized that such terms are inescapable if we want to obtain the effective action of Quantum Gravity on scales closed to the Planck length [11]. Higher-order terms in the curvature invariants, as R^2 , $R^{ij}R_{ij}$, $R^{ijkh}R_{ijkh}$, $R\Box R$, $R\Box^k R$ or non-minimally coupled terms between scalar fields and geometry, as $\phi^2 R$, have to be added to the effective Lagrangian of gravitational field when quantum corrections are considered. For example, one has to stress that such terms occur in the effective Lagrangian of strings or in Kaluza-Klein theories, when the mechanism of dimensional reduction is used [12].

From a conceptual point of view, there would be no *a priori* reason to restrict the gravitational Lagrangian to a linear function of the Ricci scalar R , minimally coupled with matter [13]. The idea that there are no “exact” laws of physics but that the Lagrangians of physical interactions are “stochastic” functions – with the property that local gauge invariances (*i.e.* conservation laws) are well approximated in the low energy limit and that physical constants can vary – has been taken into serious consideration – see Ref. [14] and references therein.

Besides fundamental physics motivations, all these theories have acquired a huge interest in cosmology due to the fact that they “naturally” exhibit inflationary behaviors able to overcome the shortcomings of Standard Cosmological Model (based on GR). The related cosmological models seem very realistic and capable of matching with the observations [15, 16, 17].

Furthermore, it is possible to show that, via conformal transformations, the higher-order and non-minimally coupled terms always correspond to Einstein gravity plus one or more than one minimally coupled scalar fields [18, 19, 20,

21, 22]. More precisely, higher-order terms always appear as a contribution of order two in the equations of motion. For example, a term like R^2 gives fourth order equations [23], $R \square R$ gives sixth order equations [21, 24], $R \square^2 R$ gives eighth order equations [25] and so on. By a conformal transformation, any 2nd-order of derivation corresponds to a scalar field: for example, fourth-order gravity gives Einstein plus one scalar field, sixth order gravity gives Einstein plus two scalar fields and so on [21, 26]. This feature results very interesting if we want to obtain multiple inflationary events since a former early stage could select “very” large-scale structures (clusters of galaxies today), while a latter stage could select “small” large-scale structures (galaxies today) [24]. The philosophy is that each inflationary era is connected with the dynamics of a scalar field. Furthermore, these extended schemes naturally could solve the problem of “graceful exit” bypassing the shortcomings of former inflationary models [16, 27].

Recently, ETG are going also to play an interesting role to describe the today observed Universe. In fact, the amount of good quality data of last decade has made it possible to shed new light on the effective picture of the Universe. Type Ia Supernovae (SNeIa) [28], anisotropies in the cosmic microwave background radiation (CMBR) [29], and matter power spectrum inferred from large galaxy surveys [30] represent the strongest evidences for a radical revision of the Cosmological Standard Model also at recent epochs. In particular, the *concordance* Λ CDM model predicts that baryons contribute only for $\sim 4\%$ of the total matter-energy budget, while the exotic *cold dark matter* (CDM) represents the bulk of the matter content ($\sim 25\%$) and the cosmological constant Λ plays the role of the so called “dark energy” ($\sim 70\%$) [31]. Although being the best fit to a wide range of data [32], the Λ CDM model is severely affected by strong theoretical shortcomings [33] that have motivated the search for alternative models [34, 35].

Dark energy models mainly rely on the implicit assumption that Einstein’s GR is the correct theory of gravity indeed. Nevertheless, its validity at the larger astrophysical and cosmological scales has never been tested [36], and it is therefore conceivable that both cosmic speed up and dark matter represent signals of a breakdown in our understanding of gravitation law so that one should consider the possibility that the Hilbert-Einstein Lagrangian, linear in the Ricci scalar R , should be generalized.

Following this line of thinking, the choice of a generic function $f(R)$ can be derived by matching the data and by the “economic” requirement that no exotic ingredients have to be added. This is the underlying philosophy of what is referred to as $f(R)$ gravity [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]. In this context, the same cosmological constant could be removed as an ingredient of the cosmic pie being nothing else but a particular eigenvalue of a general class of theories [48].

However $f(R)$ gravity can be encompassed in the ETGs being a “minimal” extension of GR where (analytical) functions of Ricci scalar are taken into account.

Although higher order gravity theories have received much attention in cosmology, since they are naturally able to give rise to accelerating expansions (both in the late and in the early Universe) and systematic studies of the phase space of solutions are in progress [49, 50, 51, 52, 53], it is possible to demonstrate that $f(R)$ theories can also play a major role at astrophysical scales. In fact, modifying the gravity Lagrangian can affect the gravitational potential in the low energy limit. Provided that the modified potential reduces to the Newtonian one on the Solar System scale, this implication could represent an intriguing opportunity rather than a shortcoming for $f(R)$ theories (see for example [54, 55, 56, 57, 58]).

Furthermore, a corrected gravitational potential could offer the possibility to fit galaxy rotation curves without the need of dark matter [59, 60, 61, 62]. In addition, it is possible to work out a formal analogy between the corrections to the Newtonian potential and the usually adopted dark matter models. In general, any relativistic theory of gravitation can yield corrections to the Newton potential [63]) which, in the post-Newtonian (PPN) formalism, could give rise to tests for the same theory [36, 64, 65, 66].

In this paper, we want to face the problem to study $f(R)$ gravity considering also torsion. Torsion theories have been taken into account firstly by Cartan and then where introduced by Sciama and Kibble in order to deal with spin in General Relativity (see [67] for a review). Being the spin as fundamental as the mass of the particles, torsion was introduced in order to complete the following scheme: the mass (energy) as the source of curvature and the spin as the source of torsion.

Up to some time ago, torsion did not seem to produce models with observable effects since phenomena implying spin and gravity were considered to be significant only in the very early Universe. After, it has been proven that spin is not the only source of torsion. As a matter of fact, torsion field can be decomposed in three irreducible tensors, with different properties. In [68], a systematic classification of these different types of torsion and their possible sources is discussed. This means that a wide class of torsion models could be investigate independently of spin as their source.

In principle, torsion could be constrained at every astrophysical scale and, as recently discussed, data coming from Gravity Probe B could contribute to this goal also at Solar System level [69].

In [70, 71], a systematic discussion of metric-affine $f(R)$ gravity has been pursued. In particular, the role of connection in presence of matter has been studied considering the various possible matter actions depending on connection. The main result of these papers has been the evidence that matter can tell to spacetime how to curve as well as how to twirl.

In this paper, following the same philosophy, we want to show that, starting from a generic $f(R)$ theory, the curvature and the torsion can give rise to an effective curvature-torsion stress-energy tensor capable, in principle, to address the problem of the Dark Side of the Universe in a very general geometric scheme. We do not consider the possible microscopic distribution of spin but a general torsion vector field in $f(R)$ gravity.

The layout of the paper is the following. In Secs.II and III, we derive the metric-affine field equations of $f(R)$ gravity with torsion in empty space and in presence of matter, respectively. Sec.IV is devoted to the discussion of the formal equivalence with scalar-tensor theories, while applications to Friedmann-Robertson-Walker (FRW) cosmology are discussed in Sec.V. Summary and conclusions are drawn in Sec. VI.

II. FIELD EQUATIONS IN EMPTY SPACE

Let us discuss the main features of a $f(R)$ gravity considering the most general case in which torsion is present in a \mathcal{U}_4 manifold. In a metric-affine formulation, the metric g and the connection Γ can be, in general, considered independent fields. More precisely, the dynamical fields are pairs (g, Γ) consisting of a pseudo-Riemannian metric g and a metric compatible linear connection Γ on the space-time manifold M . The corresponding field equations are derived by varying separately with respect to the metric and the connection the action functional

$$\mathcal{A}(g, \Gamma) = \int \sqrt{|g|} f(R) ds \quad (1)$$

where f is a real function, $R(g, \Gamma) = g^{ij} R_{ij}$ (with $R_{ij} := R^h_{ihj}$) is the scalar curvature associated with the connection Γ and $ds := dx^1 \wedge \dots \wedge dx^4$. Throughout the paper we use the index notation

$$R^h_{kij} = \frac{\partial \Gamma_{jk}^h}{\partial x^i} - \frac{\partial \Gamma_{ik}^h}{\partial x^j} + \Gamma_{ip}^h \Gamma_{jk}^p - \Gamma_{jp}^h \Gamma_{ik}^p \quad (2)$$

for the curvature tensor and

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^h \frac{\partial}{\partial x^h} \quad (3)$$

for the connection coefficients.

In order to evaluate the variation $\delta \mathcal{A}$ under arbitrary deformations of the connection, we recall that, given a metric tensor g_{ij} , every metric connection Γ may be expressed as

$$\Gamma_{ij}^h = \tilde{\Gamma}_{ij}^h - K_{ij}^h \quad (4)$$

where (in the holonomic basis $\left\{ \frac{\partial}{\partial x^i}, dx^i \right\}$) $\tilde{\Gamma}_{ij}^h$ denote the coefficients of the Levi-Civita connection associated with the metric g_{ij} , while K_{ij}^h indicate the components of a tensor satisfying the antisymmetry property $K_i^{jh} = -K_i^{hj}$. This last condition ensures the metric compatibility of the connection Γ .

In view of this, we can identify the actual degrees of freedom of the theory with the (independent) components of the metric g and the tensor K . Moreover, it is easily seen that the curvature and the contracted curvature tensors associated with every connection (4) can be expressed respectively as

$$R^h_{iqj} = \tilde{R}^h_{iqj} + \tilde{\nabla}_j K_{qi}^h - \tilde{\nabla}_q K_{ji}^h + K_{ji}^p K_{qp}^h - K_{qi}^p K_{jp}^h \quad (5a)$$

and

$$R_{ij} = R^h_{ihj} = \tilde{R}_{ij} + \tilde{\nabla}_j K_{hi}^h - \tilde{\nabla}_h K_{ji}^h + K_{ji}^p K_{hp}^h - K_{hi}^p K_{jp}^h \quad (5b)$$

where \tilde{R}^h_{iqj} and $\tilde{R}_{ij} = \tilde{R}^h_{ihj}$ are respectively the Riemann and the Ricci tensors of the Levi-Civita connection $\tilde{\Gamma}$ associated with the given metric g , and $\tilde{\nabla}$ indicates the Levi-Civita covariant derivative.

Making use of the identities (5b), the action functional (1) can be written in the equivalent form

$$\mathcal{A}(g, \Gamma) = \int \sqrt{|g|} f(g^{ij} (\tilde{R}_{ij} + \tilde{\nabla}_j K_{hi}^h - \tilde{\nabla}_h K_{ji}^h + K_{ji}^p K_{hp}^h - K_{hi}^p K_{jp}^h)) ds \quad (6)$$

more suitable for variations in the connection. Taking the metric g fixed, we have the identifications $\delta\Gamma_{ij}{}^h = \delta K_{ij}{}^h$ and then the variation

$$\delta\mathcal{A} = \int \sqrt{|g|} f'(R) g^{ij} (\tilde{\nabla}_j \delta K_{hi}{}^h - \tilde{\nabla}_h \delta K_{ji}{}^h + \delta K_{ji}{}^p K_{hp}{}^h + K_{ji}{}^p \delta K_{hp}{}^h - \delta K_{hi}{}^p K_{jp}{}^h - K_{hi}{}^p \delta K_{jp}{}^h) ds \quad (7)$$

Using the divergence theorem, taking the antisymmetry properties of K into account and renaming finally some indexes, we get the expression

$$\delta\mathcal{A} = \int \sqrt{|g|} \left[-\frac{\partial f'}{\partial x^i} \delta_j^h + \frac{\partial f'}{\partial x^j} \delta_i^h + f' K_{pj}{}^p \delta_i^h - f' K_{pi}{}^p \delta_j^h - f' K_{ij}{}^h + f' K_{ji}{}^h \right] \delta K_h{}^{ij} ds \quad (8)$$

The requirement $\delta\mathcal{A} = 0$ yields therefore a first set of field equations given by

$$K_{pj}{}^p \delta_i^h - K_{pi}{}^p \delta_j^h - K_{ij}{}^h + K_{ji}{}^h = \frac{1}{f'} \frac{\partial f'}{\partial x^p} (\delta_i^p \delta_j^h - \delta_j^p \delta_i^h) \quad (9)$$

Considering that the torsion coefficients of the connection Γ are $T_{ij}{}^h := \Gamma_{ij}{}^h - \Gamma_{ji}{}^h = -K_{ij}{}^h + K_{ji}{}^h$ and thus (due to antisymmetry) $T_{pi}{}^p = -K_{pi}{}^p$, eqs. (9) can be rewritten as

$$T_{ij}{}^h + T_{jp}{}^p \delta_i^h - T_{ip}{}^p \delta_j^h = \frac{1}{f'} \frac{\partial f'}{\partial x^p} (\delta_i^p \delta_j^h - \delta_j^p \delta_i^h) \quad (10)$$

or, equivalently, as

$$T_{ij}{}^h = -\frac{1}{2f'} \frac{\partial f'}{\partial x^p} (\delta_i^p \delta_j^h - \delta_j^p \delta_i^h) \quad (11)$$

In order to study the variation $\delta\mathcal{A}$ under arbitrary deformations of the metric, it is convenient to resort to the representation (1). Indeed, from the latter, we have directly

$$\delta\mathcal{A} = \int \sqrt{|g|} \left[f'(R) R_{ij} - \frac{1}{2} f(R) g_{ij} \right] \delta g^{ij} ds \quad (12)$$

thus getting the second set of field equations

$$f'(R) R_{(ij)} - \frac{1}{2} f(R) g_{ij} = 0 \quad (13)$$

Of course, one can obtain the same equations (13) starting from the representation (6) instead of (1). In that case, the calculations are just longer.

As a remark concerning eqs. (13), it is worth noticing that any connection satisfying eqs. (4) and (11) gives rise to a contracted curvature tensor R_{ij} automatically symmetric. Indeed, since the tensor K coincides necessarily with the contorsion tensor, namely

$$K_{ij}{}^h = \frac{1}{2} (-T_{ij}{}^h + T_j{}^h{}_i - T^h{}_{ij}) \quad (14)$$

from eqs. (11) we have

$$K_{ij}{}^h = \frac{1}{3} (T_j \delta_i^h - T_p g^{ph} g_{ij}) \quad (15)$$

being

$$T_i := T_{ih}{}^h = -\frac{3}{2f'} \frac{\partial f'}{\partial x^i} \quad (16)$$

Inserting eq. (15) in eq (5b), the contracted curvature tensor can be represented as

$$R_{ij} = \tilde{R}_{ij} + \frac{2}{3} \tilde{\nabla}_j T_i + \frac{1}{3} \tilde{\nabla}_h T^h g_{ij} + \frac{2}{9} T_i T_j - \frac{2}{9} T_h T^h g_{ij} \quad (17)$$

The last expression, together with eqs. (16), entails the symmetry of the indexes i and j . Therefore, in eq. (13) we can omit the symmetrization symbol and write

$$f'(R)R_{ij} - \frac{1}{2}f(R)g_{ij} = 0 \quad (18)$$

Now, considering the trace of the equation (18), we get

$$f'(R)R - 2f(R) = 0 \quad (19)$$

The latter is an identity automatically satisfied by all possible values of R only in the special case $f(R) = \alpha R^2$. In all other cases, eq.(19) represents a constraint on the scalar curvature R .

As a conclusion follows that, if $f(R) \neq \alpha R^2$, the scalar curvature R has to be constant (at least on connected domains) and coincides with a given solution value of (19). In such a circumstance, eqs.(11) imply that the torsion $T_{ij}{}^h$ has to be zero and the theory reduces to a $f(R)$ -theory without torsion. In particular, we notice that in the case $f(R) = R$, eq. (19) yields $R = 0$ and therefore eqs. (18) are equivalent to Einstein's equations in empty space $R_{ij} = 0$. On the other hand, if we assume $f(R) = \alpha R^2$, we can have non-vanishing torsion. In this case, by replacing eq. (19) in eqs. (11) and (18), we obtain field equations of the form

$$R_{ij} - \frac{1}{4}Rg_{ij} = 0 \quad (20a)$$

$$T_{ij}{}^h = -\frac{1}{2R}\frac{\partial R}{\partial x^i}\delta_j^h + \frac{1}{2R}\frac{\partial R}{\partial x^j}\delta_i^h \quad (20b)$$

Finally, making use of eq. (17) and the consequent relation

$$R = \tilde{R} + 2\tilde{\nabla}_h T^h - \frac{2}{3}T_h T^h \quad (21)$$

in eqs. (20), we can separately point out the contribution due to the metric and that due to the torsion. In fact, directly from eqs. (20a) we have

$$\tilde{R}_{ij} - \frac{1}{4}\tilde{R}g_{ij} = -\frac{2}{3}\tilde{\nabla}_j T_i + \frac{1}{6}\tilde{\nabla}_h T^h g_{ij} - \frac{2}{9}T_i T_j + \frac{1}{18}T_h T^h g_{ij} \quad (22)$$

while from the “trace” $T_i := T_{ih}{}^h = -\frac{3}{2R}\frac{\partial R}{\partial x^i}$ of eqs. (20b), we derive

$$\frac{\partial}{\partial x^i} \left(\tilde{R} + 2\tilde{\nabla}_h T^h - \frac{2}{3}T_h T^h \right) = -\frac{2}{3} \left(\tilde{R} + 2\tilde{\nabla}_h T^h - \frac{2}{3}T_h T^h \right) T_i \quad (23)$$

Eqs. (22) and (23) are the coupled field equations in vacuum for metric and torsion in the $f(R) = \alpha R^2$ gravitational theory.

III. FIELD EQUATIONS IN PRESENCE OF MATTER

The presence of matter is embodied in the action functional (1) by adding to the gravitational Lagrangian a suitable material Lagrangian density \mathcal{L}_m , namely

$$\mathcal{A}(g, \Gamma) = \int \left(\sqrt{|g|} [f(R) + \mathcal{L}_m] \right) ds \quad (24)$$

Throughout the paper we shall consider material Lagrangian density \mathcal{L}_m not containing terms depending on torsion degrees of freedom as in [71]. The physical meaning of this assumption will be discussed later. In this case, the field equations take the form

$$f'(R)R_{ij} - \frac{1}{2}f(R)g_{ij} = \Sigma_{ij} \quad (25a)$$

$$T_{ij}{}^h = -\frac{1}{2f'(R)}\frac{\partial f'(R)}{\partial x^p}(\delta_i^p \delta_j^h - \delta_j^p \delta_i^h) \quad (25b)$$

where $\Sigma_{ij} := -\frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_m}{\delta g^{ij}}$ plays the role of the energy-momentum tensor. From the trace of eq. (25a), we obtain a fundamental relation between the curvature scalar R and the trace $\Sigma := g^{ij} \Sigma_{ij}$, which is

$$f'(R)R - 2f(R) = \Sigma \quad (26)$$

(see also [72] and references therein). In what follows, we shall systematically suppose that the relation (26) is invertible and that $\Sigma \neq \text{const.}$, thus allowing to express the curvature scalar R as a suitable function of Σ , namely

$$R = F(\Sigma) \quad (27)$$

With this assumption in mind, using eqs. (26) and (27) we can rewrite equations (25a) and (25b) in the form

$$R_{ij} - \frac{1}{2}Rg_{ij} = \frac{1}{f'(F(\Sigma))} \left(\Sigma_{ij} - \frac{1}{4}\Sigma g_{ij} \right) - \frac{1}{4}F(\Sigma)g_{ij} \quad (28a)$$

$$T_{ij}{}^h = -\frac{1}{2f'(F(\Sigma))} \frac{\partial f'(F(\Sigma))}{\partial x^p} (\delta_i^p \delta_j^h - \delta_j^p \delta_i^h) \quad (28b)$$

Moreover, making use of eqs. (17) and (21), in eq. (28a) we can decompose the contracted curvature tensor and the curvature scalar in their Christoffel and torsion dependent terms, thus getting an Einstein-like equation of the form

$$\begin{aligned} \tilde{R}_{ij} - \frac{1}{2}\tilde{R}g_{ij} = & \frac{1}{f'(F(\Sigma))} \left(\Sigma_{ij} - \frac{1}{4}\Sigma g_{ij} \right) - \frac{1}{4}F(\Sigma)g_{ij} - \frac{2}{3}\tilde{\nabla}_j T_i \\ & + \frac{2}{3}\tilde{\nabla}_h T^h g_{ij} - \frac{2}{9}T_i T_j - \frac{1}{9}T_h T^h g_{ij} \end{aligned} \quad (29)$$

Now, setting

$$\varphi := f'(F(\Sigma)) \quad (30)$$

from the trace of eqs. (28b), we obtain

$$T_i := T_{ih}{}^h = -\frac{3}{2\varphi} \frac{\partial \varphi}{\partial x^i} \quad (31)$$

Therefore, substituting in eqs. (29), we end up with the final equations

$$\begin{aligned} \tilde{R}_{ij} - \frac{1}{2}\tilde{R}g_{ij} = & \frac{1}{\varphi}\Sigma_{ij} + \frac{1}{\varphi^2} \left(-\frac{3}{2}\frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j} + \varphi \tilde{\nabla}_j \frac{\partial \varphi}{\partial x^i} + \frac{3}{4}\frac{\partial \varphi}{\partial x^h} \frac{\partial \varphi}{\partial x^k} g^{hk} g_{ij} \right. \\ & \left. - \varphi \tilde{\nabla}^h \frac{\partial \varphi}{\partial x^h} g_{ij} - V(\varphi)g_{ij} \right) \end{aligned} \quad (32)$$

where we defined the effective potential

$$V(\varphi) := \frac{1}{4} [\varphi F^{-1}((f')^{-1}(\varphi)) + \varphi^2 (f')^{-1}(\varphi)] \quad (33)$$

Eqs. (32) may be difficult to solve, nevertheless we can simplify this task finding solutions for a conformally related metric. Indeed, performing a conformal transformation of the kind $\bar{g}_{ij} = \varphi g_{ij}$, eqs. (32) may be rewritten in the easier form (see, for example, [72, 75, 76])

$$\bar{R}_{ij} - \frac{1}{2}\bar{R}\bar{g}_{ij} = \frac{1}{\varphi}\Sigma_{ij} - \frac{1}{\varphi^3}V(\varphi)\bar{g}_{ij} \quad (34)$$

where \bar{R}_{ij} and \bar{R} are respectively the Ricci tensor and the Ricci scalar curvature associated with the conformal metric \bar{g}_{ij} .

Concerning the connection Γ , solution of the variational problem $\delta \mathcal{A} = 0$, from eqs. (4), (15) and (31), one gets the explicit expression

$$\Gamma_{ij}{}^h = \tilde{\Gamma}_{ij}{}^h + \frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^j} \delta_i^h - \frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^p} g^{ph} g_{ij} \quad (35)$$

We can now compare our results with those obtained for $f(R)$ theories in the Palatini formalism [44, 70, 71, 74, 75, 77, 78]. If both the theories (with torsion and Palatini-like) are considered as “metric”, in the sense that the dynamical connection Γ is not coupled with matter $\left(\frac{\delta \mathcal{L}_m}{\delta \Gamma} = 0\right)$ and it does not define parallel transport and covariant derivative in space-time, then the two approaches are completely equivalent. Indeed, in the “metric” framework, the true connection of space-time is the Levi-Civita one associated with the metric g and the role played by the dynamical connection Γ is just to generate the right Einstein-like equations of the theory. Now, surprisingly enough, our field equations (32) are identical to the Einstein-like equations derived within the Palatini formalism [75].

On the other hand, if the theories are genuinely metric-affine, then they are different even though the condition $\frac{\delta \mathcal{L}_m}{\delta \Gamma} = 0$ holds. In order to stress this point, we recall that in a metric-affine theory the role of dynamical connection is not only that of generating Einstein-like field equations but also defining parallel transport and covariant derivative in space-time. Therefore, different connections imply different space-time properties. This means that the geodesic structure and the causal structures could not obviously coincide. For a discussion on this point see [78]. Furthermore, it can be easily shown that the dynamical connection (35) differs from that derived within the Palatini formalism. Indeed the latter results to be the Levi-Civita connection $\bar{\Gamma}$ associated with the conformal metric $\bar{g} = \varphi g$ [74, 75], while clearly (35) is not. More precisely, (35) is related to $\bar{\Gamma}$ by the projective transformation

$$\bar{\Gamma}_{ij}{}^h = \Gamma_{ij}{}^h + \frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^i} \delta_j^h \quad (36)$$

which is not allowed in the present theory because, for a fixed metric g , the connection (36) is no longer metric compatible.

To conclude, we notice that eqs. (34) are deducible from an Einstein-Hilbert like action functional only under restrictive conditions. More precisely, let us suppose that the material Lagrangian depends only on the components of the metric and not on its derivatives as well as that the trace $\Sigma = \Sigma_{ij} g^{ij}$ is independent of the metric and its derivatives. Then, from the identities

$$\sqrt{|\bar{g}|} = \varphi^2 \sqrt{|g|}, \quad \frac{\partial}{\partial g^{ij}} = \frac{1}{\varphi} \frac{\partial}{\partial \bar{g}^{ij}} \quad \text{and} \quad \Sigma_{ij} = -\frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_m}{\delta g^{ij}} = -\frac{1}{\sqrt{|\bar{g}|}} \frac{\partial \mathcal{L}_m}{\partial \bar{g}^{ij}} \quad (37)$$

we have the following relation

$$\Sigma_{ij} = -\varphi \frac{1}{\sqrt{|\bar{g}|}} \frac{\partial \mathcal{L}_m}{\partial \bar{g}^{ij}} := \varphi \bar{\Sigma}_{ij} \quad (38)$$

In view of this, and being $\varphi = \varphi(\Sigma)$, it is easily seen that eqs. (34) may be derived by varying with respect to \bar{g}^{ij} the action functional

$$\bar{\mathcal{A}}(\bar{g}) = \int \left[\sqrt{|\bar{g}|} \left(\bar{R} - \frac{2}{\varphi^3} V(\varphi) \right) + \mathcal{L}_m \right] ds \quad (39)$$

Therefore, under the stated assumptions, $f(R)$ gravity with torsion in the metric framework is conformally equivalent to an Einstein-Hilbert like theory.

IV. EQUIVALENCE WITH SCALAR-TENSOR THEORIES

The above considerations directly lead to study the relations between $f(R)$ gravity with torsion and scalar-tensor theories with the aim to investigate their possible equivalence. To this end, we recall that the action functional of a (purely metric) scalar-tensor theory is

$$\mathcal{A}(g, \varphi) = \int \left[\sqrt{|g|} \left(\varphi \tilde{R} - \frac{\omega_0}{\varphi} \varphi_i \varphi^i - U(\varphi) \right) + \mathcal{L}_m \right] ds \quad (40)$$

where φ is the scalar field which, depending on the sign of the kinetic term, could assume also the role of a phantom field [73], $\varphi_i := \frac{\partial \varphi}{\partial x^i}$ and $U(\varphi)$ is the potential of φ . For $U(\varphi) = 0$ such a theory reduces to the standard Brans-Dicke theory [5]. The matter Lagrangian $\mathcal{L}_m(g_{ij}, \psi)$ is a function of the metric and some matter fields ψ ; ω_0 is the so called Brans-Dicke parameter. The field equations derived by varying with respect to the metric and the scalar field are

$$\tilde{R}_{ij} - \frac{1}{2} \tilde{R} g_{ij} = \frac{1}{\varphi} \Sigma_{ij} + \frac{\omega_0}{\varphi^2} \left(\varphi_i \varphi_j - \frac{1}{2} \varphi_h \varphi^h g_{ij} \right) + \frac{1}{\varphi} \left(\tilde{\nabla}_j \varphi_i - \tilde{\nabla}_i \varphi_j \right) - \frac{U}{2\varphi} g_{ij} \quad (41)$$

and

$$\frac{2\omega_0}{\varphi} \tilde{\nabla}_h \varphi^h + \tilde{R} - \frac{\omega_0}{\varphi^2} \varphi_h \varphi^h - U' = 0 \quad (42)$$

where $\Sigma_{ij} := -\frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_m}{\delta g^{ij}}$ and $U' := \frac{dU}{d\varphi}$.

Taking the trace of eq. (41) and using it to replace \tilde{R} in eq. (42), one obtains the equation

$$(2\omega_0 + 3) \tilde{\nabla}_h \varphi^h = \Sigma + \varphi U' - 2U \quad (43)$$

By a direct comparison, it is immediately seen that for $\omega_0 = -\frac{3}{2}$ and $U(\varphi) = \frac{2}{\varphi} V(\varphi)$ (where $V(\varphi)$ is defined as in eq. (33)), eqs. (41) become formally identical to the Einstein-like equations (32) for a $f(R)$ theory with torsion. Moreover, in such a circumstance, eq. (43) reduces to the algebraic equation

$$\Sigma + \varphi U' - 2U = 0 \quad (44)$$

relating the matter trace Σ to the scalar field φ , exactly as it happens for $f(R)$ gravity. In particular, it is a straightforward matter to verify that (under the condition $f'' \neq 0$) eq. (44) expresses *exactly* the inverse relation of (30), namely

$$\Sigma = F^{-1}((f')^{-1}(\varphi)) \quad \Leftrightarrow \quad \varphi = f'(F(\Sigma)) \quad (45)$$

being $F^{-1}(X) = f'(X)X - 2f(X)$. In fact we have

$$U(\varphi) = \frac{2}{\varphi} V(\varphi) = \frac{1}{2} [F^{-1}((f')^{-1}(\varphi)) + \varphi (f')^{-1}(\varphi)] = [\varphi (f')^{-1}(\varphi) - f((f')^{-1}(\varphi))] \quad (46)$$

so that

$$U'(\varphi) = (f')^{-1}(\varphi) + \frac{\varphi}{f''((f')^{-1}(\varphi))} - \frac{\varphi}{f''((f')^{-1}(\varphi))} = (f')^{-1}(\varphi) \quad (47)$$

and then

$$\Sigma = -\varphi U'(\varphi) + 2U(\varphi) = f'((f')^{-1}(\varphi))(f')^{-1}(\varphi) - 2f((f')^{-1}(\varphi)) = F^{-1}((f')^{-1}(\varphi)) \quad (48)$$

As a conclusion follows that, in the “metric” interpretation, $f(R)$ theories with torsion are equivalent to $\omega_0 = -\frac{3}{2}$ Brans–Dicke theories.

Of course, the above statement is not true if we regard $f(R)$ theories as genuinely metric-affine ones. Nevertheless, also in this case it is possible to prove the equivalence between $f(R)$ theories with torsion and a certain class of Brans–Dicke theories, namely $\omega_0 = 0$ Brans–Dicke theories with torsion [76].

In this regard, let us consider the action functional

$$\mathcal{A}(g, \Gamma, \varphi) = \int \left[\sqrt{|g|} (\varphi R - U(\varphi)) + \mathcal{L}_m \right] ds \quad (49)$$

where the dynamical fields are respectively a metric g_{ij} , a metric connection Γ_{ij}^k and a scalar field φ . As mentioned above, the action (49) describes a Brans–Dicke theory with torsion and parameter $\omega_0 = 0$.

The variation with respect to φ yields the field equation

$$R = U'(\varphi) \quad (50)$$

To evaluate the variations with respect to the metric and the connection we may repeat exactly the same arguments stated in the previous discussion for $f(R)$ gravity. Omitting for brevity the straightforward details, the resulting field equations are

$$T_{ij}{}^h = -\frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^p} (\delta_i^p \delta_j^h - \delta_j^p \delta_i^h) \quad (51)$$

and

$$R_{ij} - \frac{1}{2} R g_{ij} = \frac{1}{\varphi} \Sigma_{ij} - \frac{1}{2\varphi} U(\varphi) g_{ij} \quad (52)$$

Inserting the content of eq. (50) in the trace of eq. (52)

$$\frac{1}{\varphi}\Sigma - \frac{2}{\varphi}U(\varphi) + R = 0 \quad (53)$$

we obtain again an algebraic relation between Σ and φ identical to eq. (44).

Therefore, choosing as above the potential $U(\varphi) = \frac{2}{\varphi}V(\varphi)$, from (44) we get $\varphi = f'(F(\Sigma))$. In view of this, decomposing R_{ij} and R in their Christoffel and torsion dependent terms, eqs. (51) and (52) become identical to eq. (31) and (32) respectively. As mentioned previously, this fact shows the equivalence between $f(R)$ theories and $\omega_0 = 0$ -Brans-Dicke theories with torsion, in the metric-affine framework. These considerations can be extremely useful in order to give a geometrical characterization to the Brans-Dicke scalar field.

V. APPLICATIONS TO FRW COSMOLOGY

We have seen that the field equations (32) may be recast in the form (34) by performing a conformal transformation $\bar{g}_{ij} = \varphi g_{ij}$. In order to apply the above considerations to FRW cosmological models, let us suppose that Σ_{ij} is the energy-momentum tensor of a cosmological perfect fluid with a negligible pressure and energy density ρ (dust case), namely

$$\Sigma^{ij} = \rho U^i U^j \quad (54)$$

where $\rho = \rho(\tau)$ only depends on the cosmic time and U^i is the four velocity of the fluid satisfying the condition

$$g_{ij}U^iU^j = -1 \quad (55)$$

From now on, we shall suppose $\varphi > 0$ (a sufficient condition for this is $f' > 0$) so that the vector field $\bar{U}^i := \frac{U^i}{\sqrt{\varphi}}$ represents the four velocity of the fluid with respect to the conformal metric \bar{g}_{ij} , while $\bar{U}_i := \bar{U}^j \bar{g}_{ji} = \sqrt{\varphi} U_i$ denotes the corresponding dual relation. In view of this, the identity

$$\frac{1}{\varphi}\Sigma_{ij} = \frac{\rho}{\varphi}U_iU_j = \frac{1}{\varphi^2}\bar{\Sigma}_{ij} \quad (56)$$

holds, where we have defined $\bar{\Sigma}_{ij} = \rho \bar{U}_i \bar{U}_j$. Consequently eqs. (34) may be rewritten as

$$\bar{G}_{ij} = \frac{1}{\varphi^2} \left(\bar{\Sigma}_{ij} - \frac{1}{\varphi} V(\varphi) \bar{g}_{ij} \right) \quad (57)$$

where \bar{G}_{ij} is the Einstein tensor in the barred metric. We look for a FRW solution \bar{g}_{ij} of (57), being

$$d\bar{s}^2 = -dt^2 + a^2(t) [d\psi^2 + \chi^2 d\theta^2 + \chi^2 \sin^2 \theta d\phi^2] \quad (58)$$

Therefore, once a solution \bar{g}_{ij} is found, also the conformal metric $g_{ij} = \frac{1}{\varphi} \bar{g}_{ij}$ (solution of the starting equations (32)) will be of the FRW form. Indeed, the line element associated to g_{ij} is

$$ds^2 = -\frac{1}{\varphi(t)} dt^2 + \frac{a^2(t)}{\varphi(t)} [d\psi^2 + \chi^2 d\theta^2 + \chi^2 \sin^2 \theta d\phi^2] \quad (59)$$

so that, by performing the time variable transformation

$$d\tau := \frac{1}{\sqrt{\varphi(t)}} dt \quad (60)$$

it may be expressed as

$$ds^2 = -d\tau^2 + A^2(\tau) [d\psi^2 + \chi^2 d\theta^2 + \chi^2 \sin^2 \theta d\phi^2] \quad (61)$$

with $A := \frac{a}{\sqrt{\varphi}}$. The field equations (57) reduce to the Friedmann equations

$$3 \left(\frac{\dot{a}}{a} \right)^2 + \frac{3k}{a^2} = \frac{\rho}{\varphi^2} + \frac{V(\varphi)}{\varphi^3} \quad (62a)$$

and

$$2 \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{V(\varphi)}{\varphi^3} \quad (62b)$$

which are the cosmological equations arising from our theory.

For the sake of completeness, let us derive the conservation laws of the theory. The Bianchi identities of eq. (57) give

$$\bar{\nabla}_i \left(\frac{1}{\varphi^2} \bar{\Sigma}^{ij} - \frac{1}{\varphi^3} V(\varphi) \bar{g}^{ij} \right) = 0 \quad (63)$$

where $\bar{\nabla}$ denotes the covariant derivative with respect to the Levi-Civita connection associated with \bar{g}_{ij} . In FRW metric, eqs. (63) reduce to the continuity equation

$$\frac{d}{dt} \left(\frac{\rho}{\varphi^2} a^3 \right) + a^3 \frac{d}{dt} \left(\frac{V(\varphi)}{\varphi^3} \right) = 0 \quad (64)$$

which completes the cosmological dynamical system.

VI. DISCUSSION AND CONCLUSIONS

$f(R)$ gravity seems a viable approach to solve some shortcomings coming from GR, in particular problems related to quantization on curved spacetime and cosmological issues related to early Universe (inflation) and late time dark components. Besides, the scheme of GR is fully preserved and $f(R)$ can be considered a straightforward extension where the gravitational action has not to be necessarily linear in the Ricci scalar R .

In this paper, we have discussed the possibility that also the torsion field could play an important role in the dynamics being the \mathcal{U}_4 manifolds the straightforward generalization of the pseudo-Riemannian manifolds \mathcal{V}_4 (torsionless) usually adopted in GR.

As discussed above, torsion field, in the metric-affine formalism, plays a fundamental role in clarifying the relations between the Palatini and the metric approaches: it gives further degrees of freedom which contribute, together with curvature degrees of freedom, to the dynamics. The aim is to achieve a self-consistent theory where unknown ingredients as dark energy and dark matter (up to now not detected at a fundamental level) could be completely "geometrized". Torsion field assumes a relevant role in presence of standard matter since it allows to establish a definite equivalence between scalar-tensor theories and $f(R)$ gravity, also in relation to conformal transformations.

Finally, an important point deserves a further discussion in relation to the above results. Let us consider the cosmological equation (62a). In the *lhs*, it is clear that standard matter ρ and the effective cosmological constant $\Lambda_{eff} = \frac{V(\varphi)}{\varphi^3}$ play two distinct role into the dynamics: their evolution is "tuned" by the scalar field φ (i.e. $f'(R)$). The first term could be relevant for large scale clustered structures (always involving baryonic matter and dark matter), the second term can be read as dark energy. If at present epoch they are $\frac{\rho}{\varphi^2} \simeq \frac{V(\varphi)}{\varphi^3}$, this reveals a simple mechanism to explain why we are today observing $\Omega_M \simeq 0.3$ and $\Omega_\Lambda \simeq 0.7$. Furthermore, if the field φ at denominator is small (that is $f'(R)$ is small) this could be the reason why the amounts of dark energy and dark matter result huge today.

As a toy model, let us take into account the well known $f(R) = R + \alpha R^2$ theory where, obviously, $f'(R) = 1 + 2\alpha R$. As above, the matter stress-energy tensor is $\Sigma_{ij} = \rho U_i U_j$ and then Eq.(26) becomes

$$(1 + 2\alpha R)R - 2R - 2\alpha R^2 = -\rho \quad \longleftrightarrow \quad R = \rho. \quad (65)$$

We have

$$\varphi(\rho) = f'(R(\rho)) = 1 + 2\alpha\rho \quad (66)$$

and then the term $\frac{\rho}{\varphi^2}$ becomes

$$\frac{\rho}{(1+2\alpha\rho)^2} . \quad (67)$$

Let us consider now the potential term

$$V(\varphi) = \frac{1}{4} [\varphi F^{-1}((f')^{-1}(\varphi)) + \varphi^2 (f')^{-1}(\varphi)] . \quad (68)$$

Being $(f')^{-1}(\varphi) = \rho$, one has

$$\frac{1}{4} \varphi^2 (f')^{-1}(\varphi) = \frac{1}{4} (1+2\alpha\rho)^2 \frac{1}{2\alpha} (1+2\alpha\rho-1) = \frac{1}{4} (1+2\alpha\rho)^2 \rho , \quad (69)$$

and considering the relation $F^{-1}(Y) = f'(Y)K - 2f(Y)$, it is

$$\frac{1}{4} F^{-1}((f')^{-1}(\varphi)) = \frac{1}{4} F^{-1}(\rho) = -\rho . \quad (70)$$

We have also

$$\frac{1}{4} \varphi F^{-1}((f')^{-1}(\varphi)) = -\frac{(1+2\alpha\rho)\rho}{4} , \quad (71)$$

and then we conclude that

$$V(\varphi(\rho)) = \frac{\alpha\rho^2(1+2\alpha\rho)}{2} \quad (72)$$

and

$$\frac{V(\varphi(\rho))}{\varphi^3} = \frac{\alpha\rho^2}{2(1+2\alpha\rho)^2} . \quad (73)$$

These arguments show that the condition $\frac{\rho}{\varphi^2} \simeq \frac{V(\varphi)}{\varphi^3}$ can be simply achieved leading to comparable values of Ω_M and Ω_Λ . A detailed discussion of these topic, also in relation with data, will be the argument of a forthcoming paper.

-
- [1] A. Guth, *Phys. Rev. D* **23**, 347 (1981).
 - [2] S. Weinberg, *Gravitation and Cosmology*, Wiley, 1972 New York N.Y.
 - [3] I.L. Buchbinder, S.D. Odintsov, and I.L. Shapiro, *Effective Action in Quantum Gravity*, IOP Publishing (1992) Bristol.
 - [4] M. Farhoudi, *Gen. Relativ. Grav.* **38**, 1261 (2006).
 - [5] C. Brans and R.H. Dicke, *Phys. Rev.* **124**, 925 (1961).
 - [6] H. Bondi, *Cosmology*, Cambridge Univ. Press, Cambridge (1952).
 - [7] S. Capozziello, R. de Ritis, C. Rubano, and P. Scudellaro, *La Rivista del Nuovo Cimento* **4** (1996) 1.
 - [8] D.W. Sciama, *Mon. Not. R. Ast. Soc.* **113**, 34 (1953).
 - [9] V. Faraoni, *Cosmology in Scalar-Tensor Gravity*, Kluwer Academic, Dordrecht (2004).
 - [10] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge Univ. Press, Cambridge (1982).
 - [11] G. Vilkovisky, *Class. Quantum Grav.* **9**, 895 (1992).
 - [12] M. Gasperini and G. Veneziano, *Phys. Lett.* **277B** (1992) 256.
 - [13] G. Magnano, M. Ferraris, and M. Francaviglia, *Gen. Relativ. Grav.* **19**, 465 (1987).
 - [14] J. Barrow and A.C. Ottewill, *J. Phys. A: Math. Gen.* **16**, 2757 (1983).
 - [15] A.A. Starobinsky, *Phys. Lett.* **91B** (1980) 99.
 - [16] D. La and P.J. Steinhardt, *Phys. Rev. Lett.* **62** (1989) 376.
 - [17] Kerner R. 1982, *Gen. Rel. Grav.*, 14, 453.
 - [18] P. Teyssandier and Ph. Tourrenc, *J. Math. Phys.* **24**, 2793 (1983).
 - [19] K. Maeda, *Phys. Rev. D* **39**, 3159 (1989).
 - [20] S. Capozziello, R. de Ritis, A.A. Marino. *Gen. Relativ. Grav.* **30**, 1247, (1998).
 - [21] S. Gottlöber, H.-J. Schmidt, and A.A. Starobinsky, *Class. Quantum Grav.* **7**, 893 (1990).
 - [22] G. Magnano and L.M. Sokolowski, *Phys. Rev. D* **50**, 5093 (1994).

- [23] T.V. Ruzmaikina and A.A. Ruzmaikin, *JETP*, **30**, 372 (1970).
- [24] L. Amendola, A. Battaglia-Mayer, S. Capozziello, S. Gottlöber, V. Müller, F. Occhionero and H.-J. Schmidt, *Class. Quantum Grav.* **10**, L43 (1993).
- [25] A. Battaglia-Mayer and H.-J. Schmidt, *Class. Quantum Grav.* **10**, 2441 (1993).
- [26] H.-J. Schmidt, *Class. Quantum Grav.* **7**, 1023 (1990).
- [27] L. Amendola, S. Capozziello, M. Litterio, F. Occhionero, *Phys. Rev. D* **45**, 417 (1992).
- [28] Riess A.G. et al. 2004, *Astroph. Journ.*, 607, 665
- [29] Spergel D.N. et al. 2003, *Astroph. Journ. Suppl.*, 148, 175
- [30] Cole S. et al. 2005, *Mon. Not. Roy. Astron. Soc.*, 362, 505
- [31] Bahcall N.A., Ostriker J.P., Perlmutter S., Steinhardt P.J. 1999, *Science*, 284, 1481
- [32] Seljak U. et al. 2005, *Phys. Rev. D*, 71, 103515
- [33] Carroll S.M., Press W.H., Turner E.L. 1992, *Ann. Rev. Astron. Astroph.*, 30, 499
- [34] Peebles P.J.E., Rathra B. 2003, *Rev. Mod. Phys.*, 75, 559; Padmanabhan T. 2003, *Phys. Rept.*, 380, 235.
- [35] E.J. Copeland, M. Sami, S. Tsujikawa, *Int. J. Mod. Phys. D* **15**, 1753 (2006).
- [36] Will C.M. 1993, *Theory and Experiments in Gravitational Physics*, Cambridge Univ. Press, Cambridge (UK)
- [37] Capozziello S. 2002, *Int. J. Mod. Phys. D*, 11, 483
- [38] Nojiri S., Odintsov S.D., *Phys. Lett. B* **576**, 5 (2003).
- [39] Nojiri S., Odintsov S.D., *Phys. Rev. D* **68**, 123512 (2003).
- [40] Capozziello S., Carloni S., Troisi A. 2003, *Rec. Res. Dev. in Astron. and Astroph.*, 1, 1, (arXiv:astro-ph/0303041)
- [41] Carroll S.M., Duvvuri V., Trodden M., Turner M. 2004, *Phys. Rev. D*, 70, 043528.
- [42] Carroll S.M., De Felice A., Duvvuri V., Easson D.A., Trodden M. and Turner M. 2005, *Phys. Rev. D*, 71, 063513.
- [43] Flanagan E.E. 2003, *Class. Quant. Grav.*, 21, 417
- [44] Allemandi G., Borowiec A., Francaviglia M. 2004, *Phys. Rev. D*, 70, 103503
- [45] Nojiri S., Odintsov S.D., *Int. J. Geom. Meth. Mod. Phys.* **4**, 115 (2007).
- [46] Capozziello S., Cardone V.F., Troisi A. 2005, *Phys. Rev. D*, 71, 043503
- [47] T. Koivisto, *Phys. Rev. D* **73**, 083517 (2006).
- [48] S. Capozziello, R. Garattini *Class. Quant. Grav.* **24**, 1627 (2007).
- [49] Capozziello S., Nojiri S., Odintsov S.D., Troisi A., 2006, *Phys. Lett. B* (2006) 135.
- [50] L. Amendola, D. Polarski, and S. Tsujikawa, *Phys. Rev. Lett.* **98**, 131302 (2007).
- [51] L. Amendola, R. Gannouji, D. Polarski, and S. Tsujikawa, *Phys. Rev. D* **75**, 083504 (2007).
- [52] S. Carloni, P. Dunsby, S. Capozziello, A. Troisi, *Class. Quant. Grav.* **22**, 4839 (2005).
- [53] S. Carloni, A. Troisi, P.K.S. Dunsby, gr-qc/0706.0452 (2007).
- [54] J.D. Anderson et al. *Phys. Rev. Lett.* **81**, 2858 (1998); *Phys. Rev. D* **65**, 082004 (2002).
- [55] V. Faraoni, *Phys. Rev. D* **74**, 023529 (2006).
- [56] T. Faulkner et al. astro-ph/0612569 (2006).
- [57] O. Bertolami, Ch.G. Böhrer, T. Harko, F.S.N. Lobo, 0704.1733 [gr-qc] (2007).
- [58] S. Nojiri and S.D. Odintsov, arXiv: 0707.1941 [hep-th] (2007).
- [59] Capozziello S., Cardone V.F., Carloni S., Troisi A. 2004, *Phys. Lett. A*, 326, 292
- [60] Milgrom M. 1983, *Astroph. Journ.*, 270, 365; Bekenstein J. 2004, *Phys. Rev. D*, 70, 083509
- [61] S. Capozziello, V.F. Cardone and A. Troisi *Jou. Cosm. and Astrop. Phys.* **08**, 001 (2006).
- [62] Ch. Frigerio Martins and P. Salucci, to appear in *MNRAS*, ArXiv: astro-ph/0703243 (2007).
- [63] Schmidt H.J., "Lectures on mathematical cosmology," arXiv:gr-qc/0407095 (2004).
- [64] S. Capozziello, A. Troisi 2005, *Phys. Rev. D* **72**, 044022.
- [65] S. Capozziello, A. Stabile, A. Troisi, *Mod. Phys. Lett. A* **21**, (2006) 2291.
- [66] G. Allemandi, M. Francaviglia, M. Ruggiero and A. Tartaglia, *Gen. Rel. Grav* **37**, 1891 (2005).
- [67] F.W. Hehl, P. von der Heyde, G.D. Kerlick, *Rev. Mod. Phys.* **48**, 393 (1976).
- [68] S. Capozziello, G. Lambiase and C. Stornaiolo, *Ann. Phys. (Leipzig)* **10** (2001) 713.
- [69] Y. Mao, M. Tegmark, A. Guth, S. Cabi, arXiv: gr-qc/0608121 (2007).
- [70] T. P. Sotiriou and S. Liberati, *J. Phys. Conf. Ser.* **68**, 012022 (2007) [arXiv: gr-qc/0611040].
- [71] T. P. Sotiriou and S. Liberati, *Ann. Phys.* **322**, 935 (2006).
- [72] S. Capozziello and M. Francaviglia, arXiv: 0706.1146 [astro-ph] (2007).
- [73] V. Faraoni, *Class. Quant. Grav.* **22**, 3235 (2005).
- [74] T. P. Sotiriou, *Class. Quant. Grav.* **23**, 5117 (2006).
- [75] G. J. Olmo, *Phys. Rev. D* **72**, 083505 (2006).
- [76] G. Germán, *Phys. Rev. D.*, Vol. **32**, 3307 (1985).
- [77] D.N. Vollick, *Phys. Rev. D* **68**, 063510 (2003).
- [78] G. Allemandi, M. Capone, S. Capozziello, M. Francaviglia, *Gen. Relativ. Grav.* **38** (2006) 33.